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4D $\mathcal{N} = 2$ supergravity and projective superspace

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Abstract: This paper presents a projective superspace formulation for 4D $\mathcal{N} = 2$ matter-coupled supergravity. We first describe a variant superspace realization for the $\mathcal{N} = 2$ Weyl multiplet. It differs from that proposed by Howe in 1982 by the choice of the structure group $\big(\mathrm{SO}(3,1) \times \mathrm{SU}(2)$ versus $\mathrm{SO}(3,1) \times \mathrm{U}(2)\big)$, which implies that the super-Weyl transformations are generated by a covariantly chiral parameter instead of a real unconstrained one. We introduce various off-shell supermultiplets which are curved superspace analogues of the superconformal projective multiplets in global supersymmetry and which describe matter fields coupled to supergravity. A manifestly locally supersymmetric and super-Weyl invariant action principle is given. Off-shell locally supersymmetric nonlinear sigma models are presented in this new superspace.

Keywords: Extended Supersymmetry, Superspaces, Supergravity Models.

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Contents

1. Introduction

The increasing number of spinor derivatives in the superspace measure in theories with higher supersymmetry is a well-known obstacle to the construction of extended superspace actions. A resolution of the problem lies in finding invariant subspaces over which to integrate. One such setting is four-dimensional $\mathcal{N} = 2$ projective superspace [1] (a related method uses the harmonic superspace[1] of, e.g., [2, 3]). Its applications include classical sigma models and their quantization [6], as well as supersymmetric Yang-Mills theory [7, 8]. In particular, the sigma model description is well suited for the construction of new hyperkähler metrics [9, 10]. The projective supermultiplets [1, 7, 9, 11] and the action principle [1] are at the heart of this approach. For the quantum theory, it is imperative to have an off-shell formalism and extremely useful to have all symmetries manifest.

[1]Both methods make use of the superspace $\mathbb{R}^{4|8} \times \mathbb{C}P^1 = \mathbb{R}^{4|8} \times S^2$ introduced for the first time in [4]. However, they differ in (i) the structure of off-shell supermultiplets used; and (ii) the supersymmetric action principle chosen. Due to these conceptual differences, the two approaches prove to be complementary to each other in many respects. The relationship between the harmonic and projective superspace formulations is spelled out in [5].

Geometrically, projective superspace is closely connected to twistor space, a property which is being extensively studied [12].

The concept of projective superspace has also proven to be very useful for supersymmetric theories with eight supercharges in five [13] and six [14, 15] dimensions. Superconformal field theory in projective superspace has been developed in four and five dimensions [16, 17], including the formulation of general off-shell superconformal sigma models.

What has been lacking in the formalism is a description of supergravities with eight supercharges in diverse dimensions. Recently this problem has been solved in the case of five-dimensional $\mathcal{N} = 1$ matter-coupled supergravity [18 – 20]. In the present paper we develop a projective superspace formulation for four-dimensional $\mathcal{N} = 2$ supergravity and its matter couplings. In particular, we identify a suitable set of constraints which are compatible with a super-Weyl invariance and with the existence of a large family of local projective multiplets, i.e., curved space versions of the superconformal projective multiplets [17]. This allows us to elaborate a conformal supergravity setting for general $\mathcal{N} = 2$ supergravity-matter systems similar to that existing in the $\mathcal{N} = 1$ case as reviewed in, e.g., [21, 22]. Our results also include the coupling of the conformal supergravity to vector multiplets, a super Weyl-invariant action for supergravity-matter systems in projective superspace, and new formulations of off-shell locally supersymmetric nonlinear sigma models.

The paper is organized as follows: In section 2 we introduce (two) superspace formulations of the Weyl multiplet using Grimm's constraints and solution for $\mathcal{N} = 2$ supergravity [23], and comment briefly on the relation to the superspace formulation of Howe [24]. In section 3 we define the relevant projective multiplets and their transformations. Section 4 contains the coupling of the conformal supergravity multiplet to vector supermultiplets and in section 5 we formulate a locally supersymmetric and super-Weyl-invariant action in which the Lagrangian is a real projective multiplet of weight two coupled to conformal supergravity.

Before turning to the technical part of this paper, it is worth comparing the current status of superspace approaches to general matter couplings in $\mathcal{N} = 2$ supergravity to those developed long ago for $\mathcal{N} = 1$ supergravity. In the latter case, there exist two main formalisms: (i) the constrained geometric formulation mainly due to Wess and Zumino [39]; (ii) the unconstrained prepotential formulation which was presented in the most elaborated form in [25]. The approaches (i) and (ii) are intimately related, since the prepotential formulation is obtained by solving the Wess-Zumino constraints in terms of unconstrained superfields. In the case of $\mathcal{N} = 2$ supergravity, a prepotential formulation was proposed within the harmonic superspace approach twenty years ago [26, 27]. However, a relationship of this prepotential formulation to the standard curved $\mathcal{N} = 2$ superspace geometry has never been elaborated in detail (except one specific off-shell realization for $\mathcal{N} = 2$ Poincaré supergravity considered in [26]). In particular, it has never been shown how the harmonic prepotentials introduced in [27] occur as a result of solving the constraints in Howe's formulation for $\mathcal{N} = 2$ conformal supergravity [24]. On the other hand, the recent study of 5D $\mathcal{N} = 1$ supergravity [18 – 20] and the present paper clearly demonstrate that projective superspace is ideal for developing covariant geometric formulations for supergravity-matter systems with eight supercharges. Ultimately, a completely coherent superspace description of $\mathcal{N}=2$ supergravity should probably require a synthesis of the harmonic and projective superspace methods. Keeping this in mind, we intentionally use in this paper a notation consistent with both approaches.

2. Variant formulations for the $\mathcal{N}=2$ Weyl multiplet

The Weyl multiplet of four-dimensional $\mathcal{N}=2$ conformal supergravity [28−30] was realized in superspace long ago by Howe [24] (see also [31] for the earlier discussion of the superconformal aspects of $\mathcal{N}=2$ supergravity in superspace). The structure group in his approach is chosen to be $\mathrm{SO}(3,1)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, and the super-Weyl transformations are generated by a real unconstrained parameter. We have not found the formulation given in [24] to be the simplest from the point of view of the explicit off-shell construction of supergravity-matter systems. Here we present an alternative superspace formulation for $\mathcal{N}=2$ conformal supergravity. It differs from that given in [24] in the following three points: (i) the structure group is identified with $\mathrm{SO}(3,1)\times\mathrm{SU}(2)$; (ii) the geometry of curved superspace is subject to the constraints introduced by Grimm [23]; (iii) the super-Weyl transformations are generated by a covariantly chiral but otherwise unconstrained superfield. We will briefly discuss the precise correspondence between the two formulations for conformal supergravity at the end of this section.

2.1 Grimm's superspace geometry

Consider a curved 4D $\mathcal{N}=2$ superspace $\mathcal{M}^{4|8}$ parametrized by local bosonic (x) and fermionic ($\theta,\bar\theta$) coordinates $z^M=(x^m,\theta^\mu_i,\bar\theta^i_{\dot\mu})$, where $m=0,1,\cdots,3$, $\mu=1,2$, $\dot\mu=1,2$ and $i=\underline{1},\underline{2}$. The Grassmann variables θ^μ_i and $\bar\theta^i_{\dot\mu}$ are related to each other by complex conjugation: $\overline{\theta^\mu_i}=\bar\theta^{\dot\mu i}$. Following [23], the structure group is chosen to be $\mathrm{SO}(3,1)\times\mathrm{SU}(2)$, and the covariant derivative $\mathcal{D}_A=(\mathcal{D}_a,\mathcal{D}^i_\alpha,\bar{\mathcal{D}}^{\dot\alpha}_i)$ have the form

$$\begin{aligned}\mathcal{D}_A &= E_A + \Phi_A{}^{kl}\,J_{kl} + \frac{1}{2}\Omega_A{}^{bc}\,M_{bc}\\ &= E_A + \Phi_A{}^{kl}\,J_{kl} + \Omega_A{}^{\beta\gamma}\,M_{\beta\gamma}+\bar\Omega_A{}^{\dot\beta\dot\gamma}\,\bar M_{\dot\beta\dot\gamma}\ .\end{aligned}\tag{2.1}$$

Here $E_A=E_A{}^M(z)\partial_M$ is the supervielbein, with $\partial_M=\partial/\partial z^M$, $J_{kl}=J_{lk}$ are generators of the group SU(2), M_{bc} the generators of the Lorentz group $\mathrm{SO}(3,1)$, $\Phi_A{}^{kl}(z)$ and $\Omega_A{}^{bc}(z)$ the corresponding connections. The Lorentz generators with vector indices ($M_{ab}=-M_{ba}$) and spinor indices ($M_{\alpha\beta}=M_{\beta\alpha}$ and $\bar M_{\dot\alpha\dot\beta}=\bar M_{\dot\beta\dot\alpha}$) are related to each other by the rule:

$$M_{ab}=(\sigma_{ab})^{\alpha\beta}M_{\alpha\beta}-(\tilde\sigma_{ab})^{\dot\alpha\dot\beta}\bar M_{\dot\alpha\dot\beta}\ ,\quad M_{\alpha\beta}=\frac{1}{2}(\sigma^{ab})_{\alpha\beta}M_{ab}\ ,\quad \bar M_{\dot\alpha\dot\beta}=-\frac{1}{2}(\tilde\sigma^{ab})_{\dot\alpha\dot\beta}M_{ab}\ .$$

The generators of the structure group act on the covariant derivatives as follows:[2]

$$\begin{aligned}&[J_{kl},\mathcal{D}^i_\alpha]=-\delta^i_{(k}\mathcal{D}_{\alpha l)}\ ,\qquad [J_{kl},\bar{\mathcal{D}}^{\dot\alpha}_i]=-\varepsilon_{i(k}\bar{\mathcal{D}}^{\dot\alpha}_{l)}\ ,\\ &[M_{\alpha\beta},\mathcal{D}^i_\gamma]=\varepsilon_{\gamma(\alpha}\mathcal{D}^i_{\beta)}\ ,\qquad [\bar M_{\dot\alpha\dot\beta},\bar{\mathcal{D}}^i_{\dot\gamma}]=\varepsilon_{\dot\gamma(\dot\alpha}\bar{\mathcal{D}}^i_{\dot\beta)}\ ,\end{aligned}\tag{2.2}$$

[2] In what follows, the (anti)symmetrization of n indices is defined to include a factor of $(n!)^{-1}$.

while $[M_{\alpha\beta}, \bar{\mathcal{D}}^i_{\dot\gamma}] = [\bar{M}_{\dot\alpha\dot\beta}, \mathcal{D}^i_\gamma] = 0$. Our notation and conventions correspond to [22]; they almost coincide with those used in [32] except for the normalization of the Lorentz generators, including a sign definition of the sigma-matrices σ_{ab} and $\tilde{\sigma}_{ab}$.

The supergravity gauge group is generated by local transformations of the form

$$\delta_K \mathcal{D}_A = [K, \mathcal{D}_A]\,, \qquad K = K^C(z)\mathcal{D}_C + \frac{1}{2}K^{cd}(z)M_{cd} + K^{kl}(z)J_{kl}\,, \tag{2.3}$$

with the gauge parameters obeying natural reality conditions, but otherwise arbitrary. Given a tensor superfield $U(z)$, with its indices suppressed, it transforms as follows:

$$\delta_K U = K\,U\ . \tag{2.4}$$

The covariant derivatives obey (anti-)commutation relations of the form:

$$[\mathcal{D}_A, \mathcal{D}_B\} = T_{AB}{}^C\mathcal{D}_C + R_{AB}{}^{kl}J_{kl} + \frac{1}{2}R_{AB}{}^{cd}M_{cd}\,, \tag{2.5}$$

where $T_{AB}{}^C$ is the torsion, and $R_{AB}{}^{kl}$ and $R_{AB}{}^{cd}$ constitute the curvature. Following [23], some components of the torsion are subject to the following constraints:

$$T^{i}_{\alpha}{}^{\dot\beta}_{j}{}^{c} = -2\mathrm{i}\delta^i_j(\sigma^c)_\alpha{}^{\dot\beta}\,, \qquad T^{i}_{\alpha}{}^{j}_{\beta}{}^{c} = T_{i}^{\dot\alpha}{}_{j}^{\dot\beta}{}^{c} = 0 \qquad (\text{dim } 0) \tag{2.6a}$$

$$T^{i}_{\alpha}{}^{j}_{\beta}{}^{\gamma}_{k} = T^{i}_{\alpha}{}^{j}_{\beta}{}^{k}_{\dot\gamma} = T^{i}_{\alpha}{}^{\dot\beta}_{j}{}^{\gamma}_{k} = T^{i}_{\alpha}{}^{\dot\beta}_{j}{}^{k}_{\dot\gamma} = T_{i}^{\dot\alpha}{}_{j}^{\dot\beta}{}^{k}_{\dot\gamma} = T^{i}_{\alpha}{}_{b}{}^{c} = T_{i}^{\dot\alpha}{}_{b}{}^{c} = 0 \qquad (\text{dim } \tfrac{1}{2}) \tag{2.6b}$$

$$T_{a}{}^{j}_{\beta}{}^{\gamma}_{k} = \delta^j_k\,T_{a\beta}{}^{\gamma}\,, \qquad T_{a}{}^{\dot\beta}_{j}{}^{k}_{\dot\gamma} = \delta^k_j\,T_a{}^{\dot\beta}{}_{\dot\gamma}\,, \qquad T_{ab}{}^c = 0 \qquad (\text{dim } 1)\ . \tag{2.6c}$$

The solution to the constraints was given in [23]. Modulo trivial re-definitions, it is:

$$\begin{aligned}
\{\mathcal{D}^i_\alpha, \mathcal{D}^j_\beta\} &= 4S^{ij}M_{\alpha\beta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}Y^{\gamma\delta}M_{\gamma\delta} + 2\varepsilon^{ij}\varepsilon_{\alpha\beta}\bar{W}^{\dot\gamma\dot\delta}\bar{M}_{\dot\gamma\dot\delta}\\
&\quad +2\varepsilon_{\alpha\beta}\varepsilon^{ij}S^{kl}J_{kl} + 4Y_{\alpha\beta}J^{ij}\,, &(2.7\text{a})\\
\{\bar{\mathcal{D}}^{\dot\alpha}_i, \bar{\mathcal{D}}^{\dot\beta}_j\} &= -4\bar{S}_{ij}\bar{M}^{\dot\alpha\dot\beta} - 2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\bar{Y}^{\dot\gamma\dot\delta}\bar{M}_{\dot\gamma\dot\delta} - 2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}W^{\gamma\delta}M_{\gamma\delta}\\
&\quad -2\varepsilon_{ij}\varepsilon^{\dot\alpha\dot\beta}\bar{S}^{kl}J_{kl} - 4\bar{Y}^{\dot\alpha\dot\beta}J_{ij}\,, &(2.7\text{b})\\
\{\mathcal{D}^i_\alpha, \bar{\mathcal{D}}^{\dot\beta}_j\} &= -2\mathrm{i}\delta^i_j(\sigma^c)_\alpha{}^{\dot\beta}\mathcal{D}_c + 4\delta^i_jG^{\delta\dot\beta}M_{\alpha\delta} + 4\delta^i_jG_{\alpha\dot\gamma}\bar{M}^{\dot\gamma\dot\beta} + 8G_\alpha{}^{\dot\beta}J^i{}_j\,, &(2.7\text{c})\\
[\mathcal{D}_a, \mathcal{D}^j_\beta] &= \mathrm{i}(\sigma_a)_{(\beta}{}^{\dot\beta}G_{\gamma)\dot\beta}\mathcal{D}^{\gamma j} + \frac{\mathrm{i}}{2}\Big((\sigma_a)_{\beta\dot\gamma}S^{jk} - \varepsilon^{jk}(\sigma_a)_\beta{}^{\dot\delta}\bar{W}_{\dot\delta\dot\gamma} - \varepsilon^{jk}(\sigma_a)^\alpha{}_{\dot\gamma}Y_{\alpha\beta}\Big)\bar{\mathcal{D}}^{\dot\gamma}_k\\
&\quad +\frac{\mathrm{i}}{2}\Big((\sigma_a)_\beta{}^{\dot\delta}T_{cd}{}^{j}_{\dot\delta} + (\sigma_c)_\beta{}^{\dot\delta}T_{ad}{}^{j}_{\dot\delta} - (\sigma_d)_\beta{}^{\dot\delta}T_{ac}{}^{j}_{\dot\delta}\Big)M^{cd}\\
&\quad +\frac{\mathrm{i}}{2}\Big((\tilde\sigma_a)^{\dot\gamma\gamma}\varepsilon^{j(k}\bar{\mathcal{D}}^{l)}_{\dot\gamma}Y_{\beta\gamma} - (\sigma_a)_{\beta\dot\gamma}\varepsilon^{j(k}\bar{\mathcal{D}}^{l)}_{\dot\delta}\bar{W}^{\dot\gamma\dot\delta} - \frac{1}{2}(\sigma_a)_\beta{}^{\dot\gamma}\bar{\mathcal{D}}^j_{\dot\gamma}S^{kl}\Big)J_{kl}\,, &(2.7\text{d})\\
[\mathcal{D}_a, \bar{\mathcal{D}}^{\dot\beta}_j] &= \mathrm{i}(\sigma_a)_\alpha{}^{(\dot\beta}G^{\alpha\dot\gamma)}\bar{\mathcal{D}}_{\dot\gamma j} + \frac{\mathrm{i}}{2}\Big((\tilde\sigma_a)^{\dot\beta\gamma}\bar{S}_{jk} - \varepsilon_{jk}(\sigma_a)_\alpha{}^{\dot\beta}W^{\alpha\gamma} - \varepsilon_{jk}(\sigma_a)^\gamma{}_{\dot\alpha}\bar{Y}^{\dot\alpha\dot\beta}\Big)\mathcal{D}^k_\gamma\\
&\quad +\frac{\mathrm{i}}{2}\Big((\tilde\sigma_a)_\delta{}^{\dot\beta}T_{cd}{}^{\delta}_{j} + (\sigma_c)_\delta{}^{\dot\beta}T_{ad}{}^{\delta}_{j} - (\sigma_d)_\delta{}^{\dot\beta}T_{ac}{}^{\delta}_{j}\Big)M^{cd}\\
&\quad +\frac{\mathrm{i}}{2}\Big(-(\sigma_a)^\gamma{}_{\dot\gamma}\delta^{(k}_j\mathcal{D}^{l)}_\gamma\bar{Y}^{\dot\beta\dot\gamma} - (\sigma_a)_\gamma{}^{\dot\beta}\delta^{(k}_j\mathcal{D}^{l)}_\delta W^{\gamma\delta} + \frac{1}{2}(\sigma_a)_\alpha{}^{\dot\beta}\mathcal{D}^\alpha_j\bar{S}^{kl}\Big)J_{kl}\,, &(2.7\text{e})
\end{aligned}$$

where

$$T_{ab}{}^{\gamma}_{k} = -\frac{1}{4}(\tilde\sigma_{ab})^{\dot\alpha\dot\beta}\mathcal{D}^\gamma_k\bar{Y}_{\dot\alpha\dot\beta} + \frac{1}{4}(\sigma_{ab})^{\alpha\beta}\mathcal{D}^\gamma_kW_{\alpha\beta} - \frac{1}{6}(\sigma_{ab})^{\gamma\delta}\mathcal{D}^l_\delta\bar{S}_{kl}\,, \tag{2.8a}$$

$$T_{ab}{}^{k}_{\dot\gamma} = -\frac{1}{4}(\sigma_{ab})^{\alpha\beta}\bar{\mathcal{D}}^k_{\dot\gamma}Y_{\alpha\beta} + \frac{1}{4}(\tilde\sigma_{ab})^{\dot\alpha\dot\beta}\bar{\mathcal{D}}^k_{\dot\gamma}\bar{W}_{\dot\alpha\dot\beta} - \frac{1}{6}(\tilde\sigma_{ab})_{\dot\gamma\dot\delta}\bar{\mathcal{D}}^{\dot\delta}_lS^{kl}\ . \tag{2.8b}$$

Here the real four-vector $G_{\alpha\dot\alpha}$, the complex symmetric tensors $S^{ij} = S^{ji}$, $W_{\alpha\beta} = W_{\beta\alpha}$, $Y_{\alpha\beta} = Y_{\beta\alpha}$ and their complex conjugates $\bar S_{ij} := \overline{S^{ij}}$, $\bar W_{\dot\alpha\dot\beta} := \overline{W_{\alpha\beta}}$, $\bar Y_{\dot\alpha\dot\beta} := \overline{Y_{\alpha\beta}}$ obey additional constraints implied by the Bianchi identities. They comprise the dimension 3/2 identities

$$\mathcal{D}^{(i}_\alpha S^{jk)} = \bar{\mathcal{D}}^{(i}_{\dot\alpha} S^{jk)} = 0\,, \tag{2.9a}$$

$$\bar{\mathcal{D}}^{\dot\alpha}_i W_{\beta\gamma} = 0\,, \tag{2.9b}$$

$$\mathcal{D}^i_{(\alpha} Y_{\beta\gamma)} = 0\,, \tag{2.9c}$$

$$\mathcal{D}^i_\alpha S_{ij} + \mathcal{D}^\beta_j Y_{\beta\alpha} = 0\,, \tag{2.9d}$$

$$\mathcal{D}^i_\alpha G_{\beta\dot\beta} = -\frac{1}{4}\bar{\mathcal{D}}^i_{\dot\beta} Y_{\alpha\beta} + \frac{1}{12}\varepsilon_{\alpha\beta}\bar{\mathcal{D}}_{\dot\beta j} S^{ij} - \frac{1}{4}\varepsilon_{\alpha\beta}\bar{\mathcal{D}}^{\dot\gamma i}\bar W_{\dot\beta\dot\gamma}\,, \tag{2.9e}$$

as well as the dimension 2 relation

$$\big(\mathcal{D}^i_{(\alpha}\mathcal{D}_{\beta)i} - 4Y_{\alpha\beta}\big)W^{\alpha\beta} = \big(\bar{\mathcal{D}}^{(\dot\alpha}_i\bar{\mathcal{D}}^{\dot\beta)i} - 4\bar Y^{\dot\alpha\dot\beta}\big)\bar W_{\dot\alpha\dot\beta}\,. \tag{2.10}$$

At this point, Grimm stopped his analysis in 1980 [23].

It is worth pointing out that the 4D $\mathcal{N} = 2$ anti-de Sitter superspace

$$\frac{\mathrm{OSp}(2|4)}{\mathrm{SO}(3,1)\times\mathrm{SO}(2)}$$

corresponds to a supergeometry with covariantly constant torsion (compare with the case of 5D $\mathcal{N} = 1$ anti-de Sitter superspace [33]):

$$W_{\alpha\beta} = Y_{\alpha\beta} = 0\,, \qquad G_{\alpha\dot\beta} = 0\,, \qquad \mathcal{D}^i_\alpha S^{kl} = \bar{\mathcal{D}}^i_{\dot\alpha} S^{kl} = 0\,. \tag{2.11}$$

The integrability condition for these constraints is $[S, S^\dagger] = 0$, with $S = (S^i{}_j)$, and hence $\bar S^{ij} = q\,\mathcal{S}^{ij}$, where $\bar{\mathcal{S}}^{ij} = \mathcal{S}^{ij}$ and $q \in \mathrm{U}(1)$ is a constant parameter.

2.2 Super-Weyl transformations

What was not noticed in [23], is the fact that the constraints (2.6a)–(2.6c) are invariant under super-Weyl transformations of the form:

$$\begin{aligned}
\delta_\sigma \mathcal{D}^i_\alpha &= \frac{1}{2}\bar\sigma\mathcal{D}^i_\alpha + (\mathcal{D}^{\gamma i}\sigma)M_{\gamma\alpha} - (\mathcal{D}_{\alpha k}\sigma)J^{ki}\,,\\
\delta_\sigma \bar{\mathcal{D}}_{\dot\alpha i} &= \frac{1}{2}\sigma\bar{\mathcal{D}}_{\dot\alpha i} + (\bar{\mathcal{D}}^{\dot\gamma}_i\bar\sigma)\bar M_{\dot\gamma\dot\alpha} + (\bar{\mathcal{D}}^k_{\dot\alpha}\bar\sigma)J_{ki}\,,\\
\delta_\sigma \mathcal{D}_a &= \frac{1}{2}(\sigma+\bar\sigma)\mathcal{D}_a + \frac{\mathrm{i}}{4}(\sigma_a)^\alpha{}_{\dot\beta}(\mathcal{D}^k_\alpha\sigma)\bar{\mathcal{D}}^{\dot\beta}_k + \frac{\mathrm{i}}{4}(\sigma_a)^\alpha{}_{\dot\beta}(\bar{\mathcal{D}}^{\dot\beta}_k\bar\sigma)\mathcal{D}^k_\alpha - \frac{1}{2}\big(\mathcal{D}^b(\sigma+\bar\sigma)\big)M_{ab}\,,
\end{aligned} \tag{2.12}$$

where the parameter σ is an arbitrary covariantly chiral superfield,

$$\bar{\mathcal{D}}_{\dot\alpha i}\sigma = 0\,. \tag{2.13}$$

The dimension-1 components of the torsion transform under (2.12) as follows:

$$\delta_\sigma S^{ij} = \bar\sigma S^{ij} - \frac{1}{4}\mathcal{D}^{\gamma(i}\mathcal{D}^{j)}_\gamma\sigma\,, \tag{2.14a}$$

$$\delta_\sigma Y_{\alpha\beta} = \bar\sigma Y_{\alpha\beta} - \frac{1}{4}\mathcal{D}^k_{(\alpha}\mathcal{D}_{\beta)k}\sigma\,, \tag{2.14b}$$

$$\delta_\sigma W_{\alpha\beta} = \sigma W_{\alpha\beta}\,, \tag{2.14c}$$

$$\delta_\sigma G_{\alpha\dot\beta} = \frac{1}{2}(\sigma+\bar\sigma)G_{\alpha\dot\beta} - \frac{\mathrm{i}}{4}\mathcal{D}_{\alpha\dot\beta}(\sigma-\bar\sigma)\,. \tag{2.14d}$$

Observe that the covariantly chiral bi-spinor $W_{\alpha\beta}$ transforms homogeneously, and therefore it is a superfield extension of the Weyl tensor, and that the θ-independent component of G_a, $V_a(x) := G_a|$, transforms as a gauge field with respect to the local chiral rotations generated by $\lambda(x) := -\frac{\mathrm{i}}{2}(\sigma - \bar{\sigma})|$. Here the notation is that $U| := U(x,\theta,\bar{\theta})|_{\theta=\bar{\theta}=0}$, with $U(x,\theta,\bar{\theta})$ an arbitrary superfield.

Using super-Weyl transformations, one can gauge away $S^{ij}|$, $Y_{\alpha\beta}|$ and some higher-order components of these tensors. Actually, using both the supergravity gauge transformations and the super-Weyl ones, one can choose a Wess-Zumino gauge in which the surviving component fields match exactly those in the Weyl multiplet [28] except one field usually included in the Weyl multiplet — the gauge field of dilatations, $b_m(x)$. However, the latter is merely a cosmetic feature of the superconformal tensor calculus, and has no dynamical impact, as it can be algebraically gauged away by local special conformal transformations. We hope to discuss these issues in more detail in a separate publication in which the supersymmetric action (5.2) will be reduced to components in the Wess-Zumino gauge. Actually, there is a simpler independent way to justify the claim that the above superspace setting describes the Weyl multiplet. As will be argued in subsection 2.4, our formulation corresponds to a partial gauge fixing in Howe's formulation for $\mathcal{N} = 2$ conformal supergravity [24]. Such a gauge fixing eliminates only component fields which can algebraically be gauged away. Therefore, the superspace setting presented is adequate to describe the Weyl multiplet.

2.3 Reduced formulation

The super-Weyl gauge freedom can be used to gauge away the real or the imaginary part of S^{ij}. For concreteness, let us choose the first option and impose the gauge condition

$$S^{ij} = \mathrm{i}\mathcal{S}^{ij} , \qquad \bar{S}^{ij} = \mathcal{S}^{ij} . \tag{2.15}$$

Then, the residual super-Weyl transformations are generated by a covariantly chiral parameter σ constrained as follows:

$$\big(\mathcal{D}^{\alpha(i}\mathcal{D}_\alpha^{j)} + 4\mathrm{i}\,\mathcal{S}^{ij}\big)\sigma = -\big(\bar{\mathcal{D}}_{\dot{\alpha}}^{(i}\bar{\mathcal{D}}^{\dot{\alpha}j)} - 4\mathrm{i}\,\mathcal{S}^{ij}\big)\bar{\sigma} . \tag{2.16}$$

Such a setting is also adequate to describe the Weyl multiplet.

2.4 Comments on Howe's formulation

As mentioned earlier, the structure group in Howe's formulation for $\mathcal{N} = 2$ conformal supergravity [24] is $\mathrm{SO}(3,1)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. The constraints on the geometry of superspace, which were postulated in [24], are invariant under super-Weyl transformations generated by a real unconstrained superfield U. The general solution to the constraints involves more irreducible components for the torsion than the set given in section 2. The main difference from Grimm's formulation [23] is that in [24] there occurs an additional tensor $G^{ij}_{\alpha\dot{\alpha}} = G^{ji}_{\alpha\dot{\alpha}}$, along with the vector $G_{\alpha\dot{\alpha}}$ present in [23]. The super-Weyl transformations act on $G^{ij}_{\alpha\dot{\alpha}}$ according to

$$\delta G^{ij}_{\alpha\dot{\alpha}} \;=\; U\, G^{ij}_{\alpha\dot{\alpha}} + c\,[\mathcal{D}_\alpha^{(i}, \bar{\mathcal{D}}_{\dot{\alpha}}^{j)}]U , \tag{2.17}$$

for some non-zero numerical coefficient c. The constraints are such that $G^{ij}_{\alpha\dot\alpha}$ can be gauged away by super-Weyl transformations. Then, it can be shown that the U(1) connection can completely be gauged away by corresponding U(1)-gauge transformations. In the gauge $G^{ij}_{\alpha\dot\alpha} = 0$, the residual super-Weyl freedom is described by a parameter constrained by

$$[\mathcal{D}^{(i}_\alpha, \bar{\mathcal{D}}^{j)}_{\dot\alpha}]U = 0 \quad \Longrightarrow \quad U = \frac{1}{2}(\sigma + \bar\sigma)\,, \qquad \bar{\mathcal{D}}_{\dot\alpha i}\sigma = 0\;. \tag{2.18}$$

In this super-Weyl gauge, Howe's formulation reduces to that described in section 2. The action for conformal supergravity in superspace is [34]

$$S = \int \mathrm{d}^4x\, \mathrm{d}^4\Theta\, \mathcal{E}\, W^{\alpha\beta} W_{\alpha\beta} \;+\; \mathrm{c.c.}\,, \tag{2.19}$$

with $\mathcal{E}(x, \Theta)$ the chiral density,[3] is super-Weyl invariant before imposing the super-Weyl gauge (2.18). Therefore, upon fixing this gauge, the action remains invariant under the restricted super-Weyl transformations (2.12), with σ covariantly chiral.

The above picture is completely analogous to the situation in 4D $\mathcal{N} = 1$ supergravity. To describe the multiplet of conformal supergravity in superspace, one can introduce a set of constraints that are invariant under super-Weyl transformations generated by a complex unconstrained superfield L [37] (see [22] for a review). The torsion components are given in terms of a spinor superfield T_α, chiral superfields R and $W_{(\alpha\beta\gamma)}$, and a real vector $G_{\alpha\dot\alpha}$. The super-Weyl transformations can be used to gauge away T_α. In the gauge $T_\alpha = 0$, one stays with a residual super-Weyl invariance described by $L = \frac{1}{2}\sigma - \bar\sigma$, with σ a covariantly chiral superfield [38]. The resulting formulation, which is known as the old minimal formulation of $\mathcal{N} = 1$ supergravity [39], is perfectly suited to describe $\mathcal{N} = 1$ conformal supergravity. It is much easier to work with than the original formulation.

3. Projective supermultiplets

Before introducing an important family of covariant multiplets in curved superspace, it is worth recalling the definition of rigid projective superfields [1, 7, 9, 11].

3.1 Review of rigid projective superspace

In flat global $\mathcal{N} = 2$ superspace $\mathbb{R}^{4|8}$ parametrized by $z^A = (x^a, \theta^\alpha_i, \bar\theta^i_{\dot\alpha})$, the spinor covariant derivatives obey the algebra:

$$\{D^i_\alpha, D^j_\beta\} = \{\bar D_{\dot\alpha i}, \bar D_{\dot\beta j}\} = 0\,, \qquad \{D^i_\alpha, \bar D_{\dot\beta j}\} = -2\mathrm{i}\,\delta^i_j(\sigma^c)_{\alpha\dot\beta}\,\partial_c\;. \tag{3.1}$$

Making use of an isotwistor $u^+_i \in \mathbb{C}^2 \setminus \{0\}$ one may introduce a subset of spinor covariant derivatives $D^+_\alpha := D^i_\alpha\, u^+_i$ and $\bar D^+_{\dot\alpha} := \bar D^i_{\dot\alpha}\, u^+_i$ that are linear *holomorphic* functions of u^+ and strictly anticommute

$$\{D^+_\alpha, D^+_\beta\} \;=\; \{\bar D^+_{\dot\alpha}, \bar D^+_{\dot\beta}\} = \{D^+_\alpha, \bar D^+_{\dot\beta}\} = 0\;. \tag{3.2}$$

[3]Here the Grassmann variables Θ's, which are used to parametrize covariantly chiral superfields and chiral densities, were introduced in [35, 36], see [34] for a review.

A projective superfield $Q(z,u^+)$ is defined to obey the constraints $D^+_\alpha Q = \bar D^+_{\dot\alpha} Q = 0$ and be a *holomorphic homogeneous* function of u^+, $Q(z,c\,u^+) = c^n\,Q(z,u^+)$, with $c \in \mathbb{C}^* := \mathbb{C}\setminus\{0\}$, living on an open domain of $\mathbb{C}^2\setminus\{0\}$. Thus, the isotwistor $u^+_i \in \mathbb{C}^2\setminus\{0\}$ appears to be defined modulo the equivalence relation $u^+_i \sim c\,u^+_i$, with $c\in\mathbb{C}^*$, hence the superfields introduced live on projective superspace $\mathbb{R}^{4|8}\times\mathbb{C}P^1$. The projective multiples are holomorphic with respect to a local complex coordinate ζ used to parametrize $\mathbb{C}P^1$. In the north chart of $\mathbb{C}P^1$, where $u^{+\underline{1}}\neq 0$, this coordinate can be defined in the standard way: $u^{+i} = u^{+\underline{1}}(1,\zeta)$.

3.2 A projective superspace for supergravity

We consider curved 4D $\mathcal{N}=2$ superspace, in complete analogy with the case of 5D $\mathcal{N}=1$ supergravity [18–20]: that is, we view the isotwistor variables $u^+_i\in\mathbb{C}^2\setminus\{0\}$ to be local coordinates that are *inert* with respect to the subgroup SU(2) of the supergravity gauge group. The reason for doing this is that it allows us to keep the coordinates u^+_i constant; if they transformed under the local SU(2) gauge symmetry, $\mathcal{D}^i_\alpha u^+_j$ could not vanish because of the form of the constraints (2.7a). For most applications, it is sufficient to work with a large family of the isotwistor superfields, $U^{(n)}(z,u^+)$, which are described in detail in the appendix and which possess well-defined transformation laws with respect to the supergravity gauge group. It is important to note that since the u^+_i are constant, $U^{(n)}(z,u^+)$ is *not* a scalar field. Indeed, **all equations involving u^+_i must be homogeneous in u^+_i to be covariant**. In this approach, the u^+_i serve merely to totally symmetrize all SU(2) indicies.

It might well be interesting to consider a projective superspace formalism where the u^+_i *do* transform under the gauge SU(2); in that case, we would have to find appropriate constraints to avoid introducing new degrees of freedom into the theory. We leave this for future research.

The operators $\mathcal{D}^+_\alpha := u^+_i\,\mathcal{D}^i_\alpha$ and $\bar{\mathcal{D}}^+_{\dot\alpha} := u^+_i\,\bar{\mathcal{D}}^i_{\dot\alpha}$ map the isotwistor superfields into isotwistor superfields with $\frac{1}{2}$ unit higher isospin and obey the (isospin 1) anti-commutation relations:

$$\{\mathcal{D}^+_\alpha,\mathcal{D}^+_\beta\} = 4\,Y_{\alpha\beta}\,J^{++} + 4\,S^{++}M_{\alpha\beta}\,, \qquad \{\mathcal{D}^+_\alpha,\bar{\mathcal{D}}^+_{\dot\beta}\} = 8\,G_{\alpha\dot\beta}\,J^{++}\,, \tag{3.3}$$

where $J^{++} := u^+_i u^+_j\,J^{ij}$ and $S^{++} := u^+_i u^+_j\,S^{ij}$.

A projective supermultiplet of weight n, $Q^{(n)}(z,u^+)$, is a constrained isotwistor superfield. Specifically, it is a scalar superfield that lives on $\mathcal{M}^{4|8}$, is holomorphic with respect to the isotwistor variables u^+_i on an open domain of $\mathbb{C}^2\setminus\{0\}$, and is characterized by the following conditions:
(i) it obeys the covariant analyticity constraints

$$\mathcal{D}^+_\alpha Q^{(n)} = \bar{\mathcal{D}}^+_{\dot\alpha} Q^{(n)} = 0\ ; \tag{3.4}$$

(ii) it is a homogeneous function of u^+ of degree n, that is,

$$Q^{(n)}(z,c\,u^+) \;=\; c^n\,Q^{(n)}(z,u^+)\,, \qquad c\in\mathbb{C}^*\ ; \tag{3.5}$$

(iii) gauge transformations (2.3) act on $Q^{(n)}$ as follows:

$$\begin{aligned}
\delta_K Q^{(n)} &= \Big(K^C \mathcal{D}_C + K^{ij} J_{ij}\Big) Q^{(n)} , \\
K^{ij} J_{ij} Q^{(n)} &= -\frac{1}{(u^+u^-)} \Big(K^{++} D^{--} - n\, K^{+-}\Big) Q^{(n)} , \qquad K^{\pm\pm} = K^{ij}\, u^{\pm}_i u^{\pm}_j ,
\end{aligned} \tag{3.6}$$

where

$$D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}} , \qquad D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} . \tag{3.7}$$

The transformation law (3.6) involves an additional isotwistor, u^-_i, which is subject to the only condition $(u^+u^-) := u^{+i} u^-_i \neq 0$, and is otherwise completely arbitrary. By construction, $Q^{(n)}$ is independent of u^-, i.e. $\partial Q^{(n)}/\partial u^{-i} = 0$, and hence $D^{++} Q^{(n)} = 0$. One can see that $\delta_K Q^{(n)}$ is also independent of the isotwistor u^-, $\partial(\delta_K Q^{(n)})/\partial u^{-i} = 0$, due to (3.5). It follows from (3.6)

$$J^{++}\, Q^{(n)} = 0 , \qquad J^{++} \propto D^{++} , \tag{3.8}$$

and hence the covariant analyticity constraints (3.4) are indeed consistent.

It follows from (2.9a) that

$$S^{++} := S^{ij} u^+_i u^+_j , \qquad \widetilde{S}^{++} := \bar{S}^{ij} u^+_i u^+_j \tag{3.9}$$

are projective supermultiplets of weight +2,

$$\mathcal{D}^+_\alpha S^{++} = \bar{\mathcal{D}}^+_{\dot\alpha} S^{++} = 0 . \tag{3.10}$$

Let $Q^{(n)}(z, u^+)$ be a projective supermultiplet of weight n. Assuming that it transforms homogeneously under the super-Weyl transformations, the analyticity constraints uniquely fix its transformation law:

$$\delta_\sigma Q^{(n)} = \frac{n}{2} \big(\sigma + \bar\sigma\big) Q^{(n)} . \tag{3.11}$$

The assumption of homogeneity of the transformation law is crucial for the derivation of (3.11); there are some fields, such as the torsion component S^{++}, which is a projective multiplet of weight +2, that transform inhomogeneously under the super-Weyl transformation,

$$\delta_\sigma S^{++} = \bar\sigma S^{++} - \frac{1}{4} (\mathcal{D}^+)^2 \sigma , \tag{3.12}$$

in accordance with (2.14a).

Given a projective multiplet $Q^{(n)}(z, u^+)$, its complex conjugate is not covariantly analytic. However, similar to the case of flat superspace, one can introduce a generalized, analyticity-preserving conjugation, $Q^{(n)} \to \widetilde{Q}^{(n)}$, defined as

$$\widetilde{Q}^{(n)}(u^+) \equiv \bar{Q}^{(n)}\big(\overline{u^+} \to \widetilde{u}^+\big) , \qquad \widetilde{u}^+ = \mathrm{i}\, \sigma_2\, u^+ , \tag{3.13}$$

with $\bar{Q}^{(n)}(\overline{u^+})$ the complex conjugate of $Q^{(n)}$. It is not difficult to check that $\widetilde{Q}^{(n)}(z, u^+)$ is a projective multiplet of weight n. One can see that $\widetilde{\widetilde{Q}}^{(n)} = (-1)^n Q^{(n)}$, and therefore real supermultiplets can be consistently defined when n is even. In what follows, $\widetilde{Q}^{(n)}$ will be called the smile-conjugate of $Q^{(n)}$. Geometrically, this conjugation is complex conjugation composed with the antipodal map on the projective space $\mathbb{C}P^1$.

Consider a supergravity background. The superconformal group of this space is defined to be generated by those combined infinitesimal transformations (2.3) and (2.12) which do not change the covariant derivatives,

$$\delta_K \mathcal{D}_A + \delta_\sigma \mathcal{D}_A = 0 \ . \tag{3.14}$$

This definition is analogous to that often used in 4D $\mathcal{N} = 1$ supergravity [22]. In the case of 4D $\mathcal{N} = 2$ flat superspace, it is equivalent to the definition of the superconformal Killing vectors, see [17] and references therein. In this case, the transformation laws of the projective multiplets reduce to those describing the rigid superconformal projective multiplets [17].

To gain further insight into the structure of projective supermultiplets $Q^{(n)}(z, u^+)$, it is instructive to switch from their description in terms of the homogeneous coordinates, u^+_i, for $\mathbb{C}P^1$ to a formulation that makes use of the inhomogeneous local complex variable ζ which is invariant under the projective rescalings $u^+_i \to c u^+_i$. In such a setting, one should replace $Q^{(n)}(z, u^+)$ with a new superfield $Q^{[n]}(z, \zeta) \propto Q^{(n)}(z, u^+)$, where $Q^{[n]}(z, \zeta)$ is holomorphic with respect to ζ, and its explicit definition depends on the supermultiplet under consideration. One can cover $\mathbb{C}P^1$ by two open charts in which ζ can be defined, and the simplest choice is: (i) the north chart characterized by $u^{+\underline{1}} \neq 0$; (ii) the south chart with $u^{+\underline{2}} \neq 0$. Below, our consideration will be restricted to the north chart.

In the north chart $u^{+\underline{1}} \neq 0$, the projective-invariant variable $\zeta \in \mathbb{C}$ is defined as

$$u^{+i} = u^{+\underline{1}}(1, \zeta) = u^{+\underline{1}} \zeta^i \, , \qquad \zeta^i = (1, \zeta) \, , \qquad \zeta_i = \varepsilon_{ij} \, \zeta^j = (-\zeta, 1) \ . \tag{3.15}$$

Since any projective multiplet $Q^{(n)}$ and its transformation (3.6) do not depend on u^-, we can make a convenient choice for the latter. In the north chart, it is

$$u^-_i = (1, 0) \, , \qquad u^{-i} = \varepsilon^{ij} \, u^-_j = (0, -1) \ . \tag{3.16}$$

The transformation parameters K^{++} and K^{+-} in (3.6) can be represented as $K^{++} = \big(u^{+\underline{1}}\big)^2 K^{++}(\zeta)$ and $K^{+-} = u^{+\underline{1}} K(\zeta)$, where

$$K^{++}(\zeta) = K^{ij} \, \zeta_i \zeta_j = K^{\underline{11}} \zeta^2 - 2K^{\underline{12}} \zeta + K^{\underline{22}} \, , \quad K(\zeta) = K^{\underline{1}i} \, \zeta_i = -K^{\underline{11}} \zeta + K^{\underline{12}} \ . \tag{3.17}$$

If the projective supermultiplet $Q^{(n)}(z, u^+)$ is described by $Q^{[n]}(z, \zeta) \propto Q^{(n)}(z, u^+)$ in the north chart, then the covariant analyticity conditions (3.4) becomes

$$\begin{aligned} \mathcal{D}^+_\alpha(\zeta) \, Q^{[n]}(\zeta) &= 0 \, , \qquad \mathcal{D}^+_\alpha(\zeta) = -\mathcal{D}^i_\alpha \zeta_i = \zeta \, \mathcal{D}^{\underline{1}}_\alpha - \mathcal{D}^{\underline{2}}_\alpha \, , \\ \bar{\mathcal{D}}^{+\dot{\alpha}}(\zeta) \, Q^{[n]}(\zeta) &= 0 \, , \qquad \bar{\mathcal{D}}^{+\dot{\alpha}}(\zeta) = \bar{\mathcal{D}}^{\dot{\alpha}}_i \zeta^i = \bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}} + \zeta \bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{2}} \ . \end{aligned} \tag{3.18}$$

Let us give several important examples of projective supermultiplets.

An *arctic* multiplet[4] of weight n is defined to be holomorphic on the north chart. It can be represented as

$$\Upsilon^{(n)}(z,u) = (u^{+\underline{1}})^n\, \Upsilon^{[n]}(z,\zeta)\,, \qquad \Upsilon^{[n]}(z,\zeta) = \sum_{k=0}^{\infty} \Upsilon_k(z)\zeta^k\ . \tag{3.19}$$

The transformation law of $\Upsilon^{[n]}$ can be read off from eq. (3.6) by noting (see [16, 17] for technical details)

$$K^{ij}J_{ij}\,\Upsilon^{[n]}(\zeta) = \Big(K^{++}(\zeta)\,\partial_\zeta + n\,K(\zeta)\Big)\Upsilon^{[n]}(\zeta)\,, \tag{3.20}$$

or equivalently

$$\begin{aligned} J_{\underline{11}}\Upsilon_0 = 0\,, \qquad J_{\underline{11}}\Upsilon_k &= (k-1-n)\Upsilon_{k-1}\,, \quad k>0 \\ J_{\underline{22}}\Upsilon_k &= (k+1)\Upsilon_{k+1}\,, \\ J_{\underline{12}}\Upsilon_k &= \left(\frac{n}{2}-k\right)\Upsilon_k\ . \end{aligned} \tag{3.21}$$

Eq. (3.21) defines an infinite dimensional representation of the Lie algebra su(2). It should be emphasized that the transformation of $\Upsilon^{[n]}$ preserves the functional structure of $\Upsilon^{[n]}$ defined in (3.19).

The constraints (3.18) imply

$$\begin{aligned} \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}\Upsilon_0 &= 0\,, \qquad \bar{\mathcal{D}}^{\dot\alpha}_{\underline{1}}\Upsilon_1 = -\bar{\mathcal{D}}^{\dot\alpha}_{\underline{2}}\Upsilon_0\,, \\ \mathcal{D}^{\underline{2}}_\alpha\Upsilon_0 &= 0\,, \qquad \mathcal{D}^{\underline{2}}_\alpha\Upsilon_1 = \mathcal{D}^{\underline{1}}_\alpha\Upsilon_0\ . \end{aligned} \tag{3.22}$$

The integrability conditions for these constraints can be shown to be $J_{\underline{11}}\Upsilon_0 = 0$ and $J_{\underline{11}}\Upsilon_1 = -2J_{\underline{12}}\Upsilon_0$, and they hold identically due to (3.21). Using the anticommutation relations (2.7a) and (2.7b), one can deduce from (3.22)

$$\begin{aligned} -\frac{1}{4}\Big[(\bar{\mathcal{D}}_{\underline{1}})^2 + 4\bar{S}^{\underline{22}}\Big]\Upsilon_1 &= n\,\bar{S}^{\underline{12}}\Upsilon_0\,, \\ -\frac{1}{4}\Big[(\mathcal{D}^{\underline{2}})^2 + 4S^{\underline{22}}\Big]\Upsilon_1 &= n\,S^{\underline{12}}\Upsilon_0\ . \end{aligned} \tag{3.23}$$

The smile-conjugate of $\Upsilon^{(n)}$ will be called an *antarctic* multiplet of weight n. It proves to be holomorphic on the south chart, while in the north chart it has the form

$$\widetilde{\Upsilon}^{(n)}(z,u) = (u^{+\underline{2}})^n\, \widetilde{\Upsilon}^{[n]}(z,\zeta)\,, \qquad \widetilde{\Upsilon}^{[n]}(z,\zeta) = \sum_{k=0}^{\infty}(-1)^k\bar{\Upsilon}_k(z)\frac{1}{\zeta^k}\,, \tag{3.24}$$

with $\bar{\Upsilon}_k$ the complex conjugate of Υ_k. Its transformation follows from (3.6) by noting

$$K^{ij}J_{ij}\,\widetilde{\Upsilon}^{[n]}(\zeta) = \frac{1}{\zeta^n}\Big(K^{++}(\zeta)\,\partial_\zeta + n\,K(\zeta)\Big)\Big(\zeta^n\,\widetilde{\Upsilon}^{(n)}(\zeta)\Big)\ . \tag{3.25}$$

[4] We follow the terminology introduced in the rigid supersymmetric case in [6].

The arctic multiplet $\Upsilon^{[n]}$ and its smile-conjugate $\widetilde{\Upsilon}^{(n)}$ constitute a polar multiplet.

The simplest projective supermultiplets are real $O(2n)$-multiplet, with $n = 1, 2, \ldots$

$$H^{(2n)}(z, u^+) = u^+_{i_1} \cdots u^+_{i_{2n}} H^{i_1 \cdots i_{2n}}(z)\,, \qquad \widetilde{H}^{(2n)} = H^{(2n)}\ . \tag{3.26}$$

Here the case $n = 1$ corresponds to the $\mathcal{N} = 2$ tensor multiplet [40, 41]. Such multiplets are holomorphic on $\mathbb{C}P^1$. We can represent

$$\begin{aligned} H^{(2n)}(z, u^+) &= \big(\mathrm{i}\, u^{+\underline{1}} u^{+\underline{2}}\big)^n H^{[2n]}(z, \zeta)\,, \\ H^{[2n]}(z, \zeta) &= \sum_{k=-n}^{n} H_k(z) \zeta^k\,, \qquad\qquad \bar{H}_k = (-1)^k H_{-k}\ . \end{aligned} \tag{3.27}$$

The transformation of $H^{[2n]}$ follows from (3.6) by noting

$$K^{ij} J_{ij}\, H^{[2n]} = \frac{1}{\zeta^n} \Big(K^{++}(\zeta)\, \partial_\zeta + 2n\, K(\zeta) \Big) \Big(\zeta^n H^{[2n]} \Big)\ . \tag{3.28}$$

This can be seen to be equivalent to

$$\begin{aligned} J_{\underline{11}} H_{-n} = 0\,, \qquad J_{\underline{11}} H_k &= (k - 1 - n) H_{k-1}\,, \quad -n < k \le n \\ J_{\underline{22}} H_n = 0\,, \qquad J_{\underline{22}} H_k &= (k + 1 + n) H_{k+1}\,, \quad -n \le k < n \\ J_{\underline{12}} H_k &= -k H_k\ . \end{aligned} \tag{3.29}$$

The constraints (3.18) imply

$$\begin{aligned} \bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}} H_{-n} &= 0\,, \qquad \bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}} H_{-n+1} = -\bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{2}} H_{-n}\,, \\ \mathcal{D}^{\underline{2}}_{\alpha} H_{-n} &= 0\,, \qquad \mathcal{D}^{\underline{2}}_{\alpha} H_{-n+1} = \mathcal{D}^{\underline{1}}_{\alpha} H_{-n}\,, \end{aligned} \tag{3.30}$$

and from here one deduces

$$\begin{aligned} -\frac{1}{4}\Big[(\bar{\mathcal{D}}_{\underline{1}})^2 + 4\bar{S}^{\underline{22}}\Big] H_{-n+1} &= 2n\, \bar{S}^{\underline{12}} H_{-n}\,, \\ -\frac{1}{4}\Big[(\mathcal{D}^{\underline{2}})^2 + 4S^{\underline{22}}\Big] H_{-n+1} &= 2n\, S^{\underline{12}} H_{-n}\ . \end{aligned} \tag{3.31}$$

Another important projective multiplet is a real *tropical* multiplet of weight $2n$:

$$\begin{aligned} U^{(2n)}(z, u^+) &= \big(\mathrm{i}\, u^{+\underline{1}} u^{+\underline{2}}\big)^n U^{[2n]}(z, \zeta) = \big(u^{+\underline{1}}\big)^{2n} \big(\mathrm{i}\,\zeta\big)^n U^{[2n]}(z, \zeta)\,, \\ U^{[2n]}(z, \zeta) &= \sum_{k=-\infty}^{\infty} U_k(z) \zeta^k\,, \\ \bar{U}_k &= (-1)^k U_{-k}\ . \end{aligned} \tag{3.32}$$

The SU(2)-transformation law of $U^{[2n]}(z, \zeta)$ copies (3.28). To describe a massless vector multiplet prepotential, one should choose $n = 0$. Supersymmetric real Lagrangians correspond to the choice $n = 1$, see below.

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4. Coupling to vector supermultiplets

The multiplet of conformal supergravity can naturally be coupled to off-shell vector multiplets. Let us describe in detail the case of a single Abelian vector multiplet, due to its importance for the subsequent consideration.[5] Its coupling to the Weyl multiplet is achieved, first of all, by modifying the covariant derivatives as follows:

$$\mathcal{D}_A \quad \longrightarrow \quad \boldsymbol{\mathcal{D}}_A := \mathcal{D}_A + V_A \boldsymbol{Z}\,, \tag{4.1}$$

with $V_A(z)$ the gauge connection. It is convenient to interpret the generator $\boldsymbol{Z}$ as a real central charge. In addition, one should impose appropriate covariant constraints, guided by the rigid supersymmetric formulation for the vector multiplet [42], on some components of the gauge-invariant field strength F_{AB} which appears in the algebra of gauge-covariant derivatives

$$[\boldsymbol{\mathcal{D}}_A, \boldsymbol{\mathcal{D}}_B\} = T_{AB}{}^C\, \boldsymbol{\mathcal{D}}_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^{kl} J_{kl} + F_{AB} \boldsymbol{Z}\ . \tag{4.2}$$

Here the torsion and curvature are the same as in eq. (2.5).

The components of F_{AB} are:

$$F_{\alpha\beta}^{\,i\,j} = -2\varepsilon_{\alpha\beta}\varepsilon^{ij}\bar{W}\,, \qquad F^{\dot\alpha\dot\beta}_{\,i\,j} = 2\varepsilon^{\dot\alpha\dot\beta}\varepsilon_{ij}W\,, \qquad F_{\alpha j}^{\,i\,\dot\beta} = 0\,, \tag{4.3a}$$

$$F_{a\beta}^{\ j} = \frac{\rm i}{2}(\sigma_a)_\beta{}^{\dot\gamma}\bar{\mathcal{D}}^j_{\dot\gamma}\bar{W}\,, \qquad F_{a\,j}^{\ \dot\beta} = -\frac{\rm i}{2}(\sigma_a)_\gamma{}^{\dot\beta}\mathcal{D}^\gamma_j W\,, \tag{4.3b}$$

$$\begin{aligned} F_{ab} = & -\frac{1}{8}(\sigma_{ab})_{\beta\gamma}\mathcal{D}^{\beta k}\mathcal{D}^\gamma_k W - \frac{1}{8}(\tilde\sigma_{ab})_{\dot\beta\dot\gamma}\bar{\mathcal{D}}^{\dot\beta k}\bar{\mathcal{D}}^{\dot\gamma}_k\bar{W} \\ & + \frac{1}{2}\Big((\tilde\sigma_{ab})_{\dot\alpha\dot\beta}\bar{W}^{\dot\alpha\dot\beta} - (\sigma_{ab})_{\alpha\beta}Y^{\alpha\beta}\Big)W - \frac{1}{2}\Big((\sigma_{ab})_{\alpha\beta}W^{\alpha\beta} - (\tilde\sigma_{ab})_{\dot\alpha\dot\beta}\bar{Y}^{\dot\alpha\dot\beta}\Big)\bar{W}\ . \end{aligned} \tag{4.3c}$$

Here W is a covariantly chiral superfield,

$$\bar{\mathcal{D}}_{\dot\alpha i}W = 0\,, \tag{4.4}$$

obeying the Bianchi identity

$$\Big(\mathcal{D}^{\gamma(i}\mathcal{D}^{j)}_\gamma + 4S^{ij}\Big)W = \Big(\bar{\mathcal{D}}^{(i}_{\dot\gamma}\bar{\mathcal{D}}^{j)\dot\gamma} + 4\bar{S}^{ij}\Big)\bar{W}\ . \tag{4.5}$$

Under the super-Weyl transformations, W varies as

$$\delta_\sigma W = \sigma W\ . \tag{4.6}$$

Introduce

$$\Sigma^{++} := \frac{1}{4}\Big((\mathcal{D}^+)^2 + 4S^{++}\Big)W = \Sigma^{ij}u^+_i u^+_j\ . \tag{4.7}$$

Using (4.5), one can show that Σ^{++} is a real projective supermultiplet of weight $+2$,

$$\mathcal{D}^+_\alpha\Sigma^{++} = \bar{\mathcal{D}}^+_{\dot\alpha}\Sigma^{++} = 0\,, \qquad \widetilde{\Sigma}^{++} = \Sigma^{++}\ . \tag{4.8}$$

[5] An extension to the non-Abelian case is not difficult.

The super-Weyl transformation of Σ^{++} is

$$\delta_\sigma \Sigma^{++} = \big(\sigma + \bar\sigma\big)\Sigma^{++} \,, \tag{4.9}$$

compare with (3.11).

The super-Weyl gauge freedom can be used to choose the gauge

$$W = -\mathrm{i} \,, \tag{4.10}$$

which is the flat-superspace value of the rigid central charge, see [43] for a related discussion. In this gauge, eq. (4.5) reduces to

$$S^{++} = \mathrm{i}\mathcal{S}^{++} \,, \qquad \mathcal{S}^{++} = \widetilde{\mathcal{S}}^{++} \,, \tag{4.11}$$

with $\mathcal{S}^{++}$ a real $O(2)$ multiplet. As a result, one arrives at the well-known superspace realization [24, 34] for the minimal multiplet for $\mathcal{N} = 2$ supergravity [41].

Consider now a system of several Abelian vector multiplets, and let W^μ be their covariantly chiral field strengths. Let $F(W^\mu)$ be a holomophic homogeneous function of degree one, $F(c\,W^\mu) = cF(W^\mu)$. Then, we can define a generalization of Σ^{++} (4.7):

$$\boldsymbol{\Sigma}^{++} := \frac{1}{4}\Big((\mathcal{D}^+)^2 + 4S^{++}\Big)F(W^\mu) = \boldsymbol{\Sigma}^{ij} u^+_i u^+_j \,, \qquad F(c\,W^\mu) = cF(W^\mu) \ . \tag{4.12}$$

This superfield is not real, $\boldsymbol{\Sigma}^{++} \neq \widetilde{\boldsymbol{\Sigma}}^{++}$, If F is not linear. However, it enjoys the other properties of Σ^{++} given in eqs. (4.8) and (4.9).

5. Action principle

Let $\mathcal{L}^{++}$ be a real projective multiplet of weight two. In particular, its super-Weyl transformation is

$$\delta_\sigma \mathcal{L}^{++} = \big(\sigma + \bar\sigma\big)\mathcal{L}^{++} \ . \tag{5.1}$$

Associated with $\mathcal{L}^{++}$ is the following functional

$$S = \frac{1}{2\pi}\oint (u^+ \mathrm{d}u^+) \int \mathrm{d}^4x\, \mathrm{d}^8\theta\, E\, \frac{W\bar{W}\mathcal{L}^{++}}{(\Sigma^{++})^2} \,, \qquad E^{-1} = \mathrm{Ber}(E_A{}^M) \ . \tag{5.2}$$

This functional is obviously invariant under re-scalings $u^+_i(t) \to c(t)\, u^+_i(t)$, for an arbitrary function $c(t) \in \mathbb{C}\backslash\{0\}$, where t denotes the evolution parameter along the closed integration contour. Since E is invariant under the super-Weyl transformations,

$$\delta_\sigma E = 0 \,, \tag{5.3}$$

eqs. (4.6), (4.9) and (5.1) show that S is super-Weyl invariant. The action can also be shown to be invariant under arbitrary supergravity gauge transformations, in complete analogy with the 5D considerations of [19, 20].

One can represent $\mathcal{L}^{++}$ in the form

$$\begin{aligned}\mathcal{L}^{++}(z,u^+) &= \frac{1}{16}\Big((\bar{\mathcal{D}}^+)^2+4\widetilde{S}^{++}\Big)\Big((\mathcal{D}^+)^2+4S^{++}\Big)\mathcal{U}^{(-2)}(z,u^+)\\ &= \frac{1}{16}\Big((\mathcal{D}^+)^2+4S^{++}\Big)\Big((\bar{\mathcal{D}}^+)^2+4\widetilde{S}^{++}\Big)\mathcal{U}^{(-2)}(z,u^+)\,,\end{aligned} \tag{5.4}$$

for some projective prepotential $\mathcal{U}^{(-2)}$ which is an example of the isotwistor superfields introduced in the appendix. It can be seen that $\mathcal{U}^{(-2)}$ should be inert under the super-Weyl transformations,

$$\delta_\sigma \mathcal{U}^{(-2)} = 0\,, \tag{5.5}$$

in order for $\mathcal{L}^{++}$ to possess the transformation law (5.1). Then, the action (5.2) can be rewritten as

$$S = \frac{1}{2\pi}\oint (u^+\mathrm{d}u^+)\int \mathrm{d}^4x\,\mathrm{d}^8\theta\,E\,\mathcal{U}^{(-2)}\ . \tag{5.6}$$

This relation leads to the following important result: if $\mathcal{L}^{++}$ is generated in terms of some supermultiplets to which the central charge vector multiplet does not belong, then the action S is independent of the vector multiplet chosen.

Let us demonstrate that in a flat superspace limit, eq. (5.6) is equivalent to the action principle in projective superspace [1]. Let $D_A = (\partial_a, D^i_\alpha, \bar{D}^{\dot\alpha}_i)$ be the flat covariant derivatives. We also denote by L^{++} and $U^{(-2)}$ the flat-superspace limits of $\mathcal{L}^{++}$ and $\mathcal{U}^{(-2)}$,

$$L^{++}(z,u^+) = (D^+)^4 U^{(-2)}\,, \qquad (D^+)^4 = \frac{1}{16}(\bar{D}^+)^2(D^+)^2 = \frac{1}{16}(D^+)^2(\bar{D}^+)^2\ . \tag{5.7}$$

The flat-superspace version of (5.6),

$$S_{\text{flat}} = \frac{1}{2\pi}\oint (u^+\mathrm{d}u^+)\int \mathrm{d}^4x\,\mathrm{d}^8\theta\,U^{(-2)}\,, \tag{5.8}$$

can equivalently be rewritten as

$$\begin{aligned}S_{\text{flat}} &= \frac{1}{2\pi}\oint \frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^4}\int \mathrm{d}^4x\,(D^-)^4(D^+)^4U^{(-2)}\Big|\\ &= \frac{1}{2\pi}\oint \frac{(u^+\mathrm{d}u^+)}{(u^+u^-)^4}\int \mathrm{d}^4x\,(D^-)^4L^{++}\Big|\,,\end{aligned} \tag{5.9}$$

where the spinor derivatives D^-_α and $\bar{D}^-_{\dot\alpha}$ are obtained from D^+_α and $\bar{D}^+_{\dot\alpha}$ by replacing $u^+_i \to u^-_i$, with the latter fixed (i.e. t-independent) isotwistor obeying the only constraint $(u^+(t)u^-)\neq 0$ at each point of the integration contour. This is exactly the projective superspace action [1] as reformulated in [44]. The action can be seen to be invariant under arbitrary projective transformations of the form:

$$({u_i}^-\,,\,{u_i}^+)\ \to\ ({u_i}^-\,,\,{u_i}^+)\,R\,, \qquad R = \begin{pmatrix} a & 0\\ b & c\end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})\ . \tag{5.10}$$

Without loss of generality, we can assume the north pole of $\mathbb{C}P^1$ is outside of the integration contour, hence u^+ can be represented as in eq. (3.15), with ζ the local complex coordinate for $\mathbb{C}P^1$. Using the projective invariance (5.10), we can then choose u_i^- in the form (3.16). Finally, representing L^{++} in the form

$$L^{++}(z,u^+) = \mathrm{i}\, u^{+\underline{1}} u^{+\underline{2}}\, L(z,\zeta) = \mathrm{i}\big(u^{+\underline{1}}\big)^2 \zeta\, L(z,\zeta)\,, \tag{5.11}$$

and also using the fact that L^{++} enjoys the constraints $\zeta_i D^i_\alpha L = \zeta_i \bar{D}^i_{\dot\alpha} L = 0$, we can finally rewrite $S_{\rm flat}$ as an integral over the $\mathcal{N}=1$ superspace parametrized by the following coordinates: $(x^a, \theta^\alpha_{\underline{1}}, \bar\theta^{\underline{1}}_{\dot\alpha})$. The result is

$$S_{\rm flat} = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, L\Big|_{\theta_{\underline{2}}=\bar\theta^{\underline{2}}=0}\,. \tag{5.12}$$

This is equivalent to the original form of the projective superspace action [1].

It should be pointed out that the super-Weyl gauge freedom can be fixed as in (4.10). Then, the action (5.2) becomes

$$S = \frac{1}{2\pi} \oint (u^+ \mathrm{d}u^+) \int \mathrm{d}^4x\, \mathrm{d}^8\theta\, E\, \frac{\mathcal{L}^{++}}{(\mathcal{S}^{++})^2}\,. \tag{5.13}$$

This result can be compared with the 5D $\mathcal{N}=1$ supergravity action principle [19].

The approach developed in this paper is well-suited for the off-shell description of $\mathcal{N}=2$ Poincaré supergravity and its matter couplings. Such a description only requires recasting the conceptual framework of the $\mathcal{N}=2$ superconformal tensor calculus (see [28, 30] and references therein) in our superspace setting. One should consider super-Weyl invariant couplings of the Weyl multiplet to supersymmetric matter, and then break the super-Weyl invariance. As is known, the set of matter supermultiplets should include two (conformal) compensators. One of them is universal and can be identified with the central charge vector multiplet. However, the choice of a second compensator is not unique. It can be taken to be a hypermultiplet, or a tensor multiplet, or a nonlinear multiplet. For concreteness, here we choose the first option. It is known that the action for the central charge vector multiplet can be written as a chiral integral [34]:

$$S = \frac{1}{\kappa^2} \int \mathrm{d}^4x\, \mathrm{d}^4\Theta\, \mathcal{E}\, W^2 \;+\; \text{c.c.}\,, \tag{5.14}$$

with $\mathcal{E}$ the chiral density, and κ the gravitational coupling constant. It turns out that this functional can be rewritten in the form (5.2). To achieve this, we should introduce the gauge field of the central charge vector multiplet, $\mathbb{V}(z,u^+)$, which is a real projective weight-zero superfield (tropical multiplet). Then, $\mathcal{L}^{++} \propto \mathbb{V}\,\Sigma^{++}$ and

$$\int \mathrm{d}^4x\, \mathrm{d}^4\Theta\, \mathcal{E}\, W^2 \propto \frac{1}{2\pi} \oint (u^+\mathrm{d}u^+) \int \mathrm{d}^4x\, \mathrm{d}^8\theta\, E\, \frac{W\bar{W}}{\Sigma^{++}} \mathbb{V}\,. \tag{5.15}$$

Now, let us couple the Weyl multiplet to (i) a system of Abelian vectors multiplets (including the central charge vector multiplet), with W^μ the corresponding covariantly chiral field strengths); and (ii) a system of hypermultiplets described by weight-one covariantly arctic multiplets $\Upsilon^+(z,u^+)$ and their conjugates $\widetilde{\Upsilon}^+$'s (defined in complete analogy with the 5D case [18]). The supergravity-matter Lagrangian can be chosen to be

$$\mathcal{L}^{++} = \mathbb{V}\Big(\mathbf{\Sigma}^{++} + \widetilde{\mathbf{\Sigma}}^{++}\Big) - \mathrm{i}\, K(\Upsilon^+, \widetilde{\Upsilon}^+)\,, \tag{5.16}$$

with $\mathbf{\Sigma}^{++}$ defined in (4.12), and the real function $K(\Phi,\bar{\Phi})$ obeying the homogeneity condition

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi,\bar{\Phi}) = K(\Phi,\bar{\Phi})\ . \tag{5.17}$$

The action possesses the gauge invariance

$$\delta\mathbb{V} = \lambda + \tilde{\lambda}\,, \tag{5.18}$$

with λ a weight-zero arctic multiplet. Although this invariance is not obvious, it can be established choosing a supergravity Wess-Zumino gauge and applying considerations similar to those given in the five-dimensional case [18]).

The hypermultiplet sector of (5.16) is a curved-space extension of the rigid superconformal sigma model [17] (a special family of the general $\mathcal{N}=2$ supersymmetric nonlinear sigma model [9]). Let $\mathbf{\Upsilon}^+$ be the compensator contained in our system of covariantly arctic multiplets Υ^+. By analogy with the flat case [17], we can introduce new hypermultiplet variables comprising the unique weight-one multiplet $\mathbf{\Upsilon}^+(z,u^+)$ and some set of weight-zero covariantly arctic multiplets $\upsilon^I(z,u^+)$. We can represent

$$K(\Upsilon^+, \widetilde{\Upsilon}^+) = \widetilde{\mathbf{\Upsilon}}^+ \mathbf{\Upsilon}^+\, \mathrm{e}^{-\mathcal{K}(\upsilon,\widetilde{\upsilon})}\,, \tag{5.19}$$

with $\mathcal{K}(\upsilon,\widetilde{\upsilon})$ a Kähler potential. This Lagrangian is invariant under Kähler tansformations

$$\mathbf{\Upsilon}^+ \ \longrightarrow\ \mathrm{e}^{\Lambda(\upsilon)}\, \mathbf{\Upsilon}^+\,, \qquad \mathcal{K}(\upsilon,\widetilde{\upsilon}) \ \to\ \mathcal{K}(\upsilon,\widetilde{\upsilon}) + \Lambda(\upsilon) + \bar{\Lambda}(\widetilde{\upsilon})\,, \tag{5.20}$$

with Λ a holomorphic function. Note that this is precisely the structure uncovered in [45] by considering the geometry of $\mathcal{N}=2$ supersymmetric nonlinear sigma models. The potential $K(\Upsilon^+,\widetilde{\Upsilon}^+)$ has the interpretation of the hyperkähler potential on the hyperkähler cone, and $\mathcal{K}(\upsilon,\widetilde{\upsilon})$ is the Kähler potential of the twistor space of the underlying Quaternion Kähler geometry.

6. Conclusions

In this paper we have constructed $\mathcal{N}=2$ four dimensional (conformal) supergravity in projective superspace.

Our starting point is the observation that Grimm's formulation of the superspace constraints and their solutions allow additional Weyl transformations as a symmetry. These enable us to identify the Weyl multiplet residing in Grimm's solution by going to a Wess-Zumino gauge. Equivalently, our formulation represents a partial gauge fixing of Howe's formulation of $\mathcal{N}=2$ supergravity.

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The transition to projective superspace proceeds via the introduction of isotwistor variables u_i^+ in parallel to the rigid case. An important ingredient is that these are taken to be covariantly constant, a feature which may seem at variance with covariance of the D-algebra for $u_i^+ D_\alpha^i$. However, we demonstrate explicitly that covariance is maintained when acting on isotwistor superfields. An open question for future investigations is to find a formulation where the relation between the supergravity SU(2) and the isotwistor transformations is carried by a geometric field.

Within our local projective approach we construct various matter couplings as well as a superspace action. The restriction to Poincaré supergravity is discussed.

Among the future extensions of this work we mention the derivation of the explicit $\mathcal{N}=1$ as well as $\mathcal{N}=0$ component content of our isotwistor superfields. In particular, it should be possible to compare the Poincaré supergravity content to the $\mathcal{N}=1$ formulation given in [46].

As mentioned in the introduction, it is important for the quantum theory to have a manifest off-shell formulation, and we expect that our results will find applications there.

Finally, we observed that the geometric structure of hypermultiplets coupled to supergravity described in [45] arises completely naturally.

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A. Isotwistor superfields

Consider a completely symmetric isotensor superfield, $F^{i_1\cdots i_n}(z) = F^{(i_1\cdots i_n)}(z)$. Such an object may also, in principle, carry some number of Lorentz indices, but here we are interested in its SU(2)-structure only. The gauge transformation law of $F^{i_1\cdots i_n}$ is given by eq. (2.4). In particular, the local SU(2) transformation, which is described by parameters $K^{ij} = K^{ji}$, acts on $F^{i_1\cdots i_n}$ as follows:

$$\delta_{\mathrm{SU}(2)} F^{i_1\cdots i_n} \equiv K^{kl} J_{kl}\, F^{i_1\cdots i_n} = \sum_{l=1}^{n} K^{i_l}{}_j\, F^{j i_1\cdots \widehat{i_l}\cdots i_n} = n\, F^{j(i_1\cdots i_{n-1}} K^{i_n)}{}_j\,, \qquad \text{(A.1)}$$

where the notation $\widehat{i_k}$ means that the corresponding index is missing.

It is useful to develop an alternative description for the above superfield as a holomorphic tensor field over $\mathbb{C}P^1$. With the aid of complex variables $u_i^+ \in \mathbb{C}^2 \setminus \{0\}$, following [47], let us associate with $F^{i_1\cdots i_n}(z)$ a homogeneous polynomial of u^+ of degree n defined as

$$F^{(n)}(z, u^+) = u^+_{i_1} \cdots u^+_{i_n}\, F^{i_1\cdots i_n}(z)\,, \qquad F^{(n)}(z, c\, u^+) = c^n\, F^{(n)}(z, u^+)\ . \qquad \text{(A.2)}$$

It is convenient to interpret the variables u^+_i to be homogeneous coordinates for $\mathbb{C}P^1$. The latter space emerges by factorizing $\mathbb{C}^2 \setminus \{0\}$ with respect to the equivalence relation $u^+_i \sim c\, u^+_i$, with $c \in \mathbb{C}^*$. Then, $F^{(n)}$ is known to define a holomorphic tensor field of rank $(n/2, 0)$ on $\mathbb{C}P^1$. Eq. (A.1) can now be interpreted as a transformation acting in the space of holomorphic tensor fields of rank $(n/2, 0)$ on $\mathbb{C}P^1$. It is defined as

$$\delta_{\rm SU(2)} F^{(n)}(z, u^+) := u^+_{i_1} \cdots u^+_{i_n}\, \delta_{\rm SU(2)} F^{i_1 \cdots i_n}(z) \ . \tag{A.3}$$

It turns out that this transformation law can be rewritten as follows:

$$\begin{aligned}\delta_{\rm SU(2)} F^{(n)} \equiv K^{kl} J_{kl}\, F^{(n)} &= -\frac{1}{(u^+u^-)}\Big(K^{++}D^{--} - n\, K^{+-}\Big) F^{(n)}\ , \\ K^{\pm\pm} &= K^{ij}\, u^\pm_i u^\pm_j\ ,\end{aligned} \tag{A.4}$$

with the first-order operator D^{--} defined in (3.7). The right-hand side in (A.4) involves an auxiliary complex two-vector u^-_i which is chosen to be linearly independent of u^+_i, that is $(u^+u^-) := u^{+i}u^-_i \neq 0$, but is otherwise completely arbitrary. By construction, both $F^{(n)}$ and $\delta_{\rm SU(2)} F^{(n)}$ are independent of u^-. It should be pointed out that eq. (A.4) defines the action of the covariant derivatives $\mathcal{D}_A$, eq. (2.1), on $F^{(n)}$ (for any super-vector field $\xi^A(z)$, the operator $\xi^A\, \mathcal{D}_A$ acts on the space of superfields $F^{(n)}$).

If there are two homogeneous polynomials $F^{(n)}(u^+)$ and $F^{(m)}(u^+)$, their product $F^{(n+m)}(u^+) := F^{(n)}(u^+)\, F^{(m)}(u^+)$ is a homogeneous polynomials of order $(n+m)$. In superspace, new covariant operations can be defined. Indeed, one can allow the polynomials $F^{(n)}(u^+)$ to be tensor superfields, i.e. be z-dependent and carry Lorentz indices. Then, the spinor covariant derivatives can be used to define covariant maps of $F^{(n)}$'s to $F^{(n+1)}$'s by the rule:

$$\mathcal{D}^+_\alpha F^{(n)}(z, u^+) := u^+_j u^+_{i_1} \cdots u^+_{i_n}\, \mathcal{D}^j_\alpha F^{i_1 \cdots i_n}(z) = u^+_j u^+_{i_1} \cdots u^+_{i_n}\, \mathcal{D}^{(j}_\alpha F^{i_1 \cdots i_n)}(z)\ , \tag{A.5a}$$

$$\bar{\mathcal{D}}^+_{\dot\alpha} F^{(n)}(z, u^+) := u^+_j u^+_{i_1} \cdots u^+_{i_n}\, \bar{\mathcal{D}}^j_{\dot\alpha} F^{i_1 \cdots i_n}(z) = u^+_j u^+_{i_1} \cdots u^+_{i_n}\, \bar{\mathcal{D}}^{(j}_{\dot\alpha} F^{i_1 \cdots i_n)}(z)\ . \tag{A.5b}$$

The superfield $\mathcal{D}^+_\alpha F^{(n)}$ and $\bar{\mathcal{D}}^+_{\dot\alpha} F^{(n)}$ obtained are of the type $F^{(n+1)}$. Therefore, the operators $\mathcal{D}^+_\alpha$ and $\bar{\mathcal{D}}^+_{\dot\alpha}$ are covariant derivatives that send $F^{(n)}$'s to $F^{(n)}$'s. With the definitions $\mathcal{D}^+_\alpha := u^+_j\, \mathcal{D}^j_\alpha$ and $\mathcal{D}^j_\alpha = E^j_\alpha + \frac{1}{2}\, \Omega^{j\ bc}_\alpha\, M_{bc} + \Phi^{j\ kl}_\alpha\, J_{kl}$, the right-hand side in (A.5a) is actually a direct consequence of (A.3).

When acting on $F^{(n)}$, the operators $\mathcal{D}^+_\alpha$ and $\bar{\mathcal{D}}^+_{\dot\alpha}$ can be seen to obey the anticommutation relation (3.3). For instance, it follows from the definition (A.5a)

$$\{\mathcal{D}^+_\alpha, \mathcal{D}^+_\beta\} F^{(n)} = u^+_j u^+_k u^+_{i_1} \cdots u^+_{i_n}\, \{\mathcal{D}^{(j}_\alpha, \mathcal{D}^k_\beta\} F^{i_1 \cdots i_n)}\ , \tag{A.6}$$

and it only remains to apply (2.7a). Recalling the explicit action of the SU(2) generators on isospinors, eq. (2.2), for the operator $J^{++} := u^+_j u^+_k J^{jk}$ appearing in (3.3) one obtains

$$J^{++} F^{(n)} = u^+_j u^+_k u^+_{i_1} \cdots u^+_{i_n}\, J^{(jk} F^{i_1 \cdots i_n)} = 0\ . \tag{A.7}$$

In accordance with the definition of $\delta_{\rm SU(2)} F^{(n)}(z, u^+)$, eq. (A.3), the auxiliary coordinates u^+_i are inert under the local SU(2) transformations, $\delta_{\rm SU(2)} u^+_i = 0$. This is similar to the point of view adopted for the superspace coordinates z^M. These variables are chosen to be inert under the supergravity gauge transformations (2.3) and (2.4). The latter transform only the functional form of the dynamical superfields. Since u_i^+ are inert under the local SU(2) transformations, these variables are covariantly constant, $\mathcal{D}_A u_i^+ = 0$. The latter property is implied by eqs. (A.5a) and (A.5b) in conjunction with the Leibniz rule.

The example of $F^{(n)}$'s considered can naturally be extended to define more general superfields. Let us consider a superfield $U^{(n)}(z, u^+)$ (with its Lorentz indices suppressed) that lives on $\mathcal{M}^{4|8}$, is holomorphic with respect to the isotwistor variables u_i^+ on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:

(i) it is a homogeneous function of u^+ of degree n, that is,

$$U^{(n)}(z, c\,u^+) = c^n\,U^{(n)}(z, u^+)\,, \qquad c \in \mathbb{C}^*\,; \tag{A.8}$$

(ii) supergravity gauge transformations (2.3) act on $U^{(n)}$ as follows:

$$\begin{aligned} \delta_K U^{(n)} &= \Big(K^C \mathcal{D}_C + \frac{1}{2} K^{cd} M_{cd} + K^{ij} J_{ij}\Big) U^{(n)}\,, \\ K^{ij} J_{ij}\, U^{(n)} &= -\frac{1}{(u^+ u^-)} \Big(K^{++} D^{--} - n\,K^{+-}\Big) U^{(n)}\,. \end{aligned} \tag{A.9}$$

The latter relation also defines the action of the covariant derivative $\mathcal{D}_A$, eq. (2.1), on $U^{(n)}(z, u^+)$. By construction, $U^{(n)}$ is independent of u^-, i.e. $\partial U^{(n)}/\partial u^{-i} = 0$, hence $D^{++} U^{(n)} = 0$. One can check that $\delta_K U^{(n)}$ is also independent of u^-, $\partial(\delta_K U^{(n)})/\partial u^{-i} = 0$, as a consequence of (A.8). Defining

$$J^{++} = u_i^+ u_j^+ J^{ij}\,, \qquad J^{+-} = u_i^+ u_j^- J^{ij}\,, \tag{A.10}$$

eq. (A.9) implies

$$J^{++}\,U^{(n)} = 0\,, \qquad J^{+-}\,U^{(n)} = -\frac{n}{2}\,(u^+ u^-)\,U^{(n)}\,. \tag{A.11}$$

We will call $U^{(n)}(z, u^+)$ an isotwistor superfield of weight n.

Now, consider the covariant derivatives $\mathcal{D}_\alpha^+ := u_i^+\,\mathcal{D}_\alpha^i$ and $\bar{\mathcal{D}}_{\dot\alpha}^+ := u_i^+\,\bar{\mathcal{D}}_{\dot\alpha}^i$. It is evident that $\mathcal{D}_\alpha^+ U^{(n)}$ and $\bar{\mathcal{D}}_{\dot\alpha}^+ U^{(n)}$ are isotwistor superfields of weight $(n+1)$. When acting on isotwistor superfields, the operators $\mathcal{D}_\alpha^+$ and $\bar{\mathcal{D}}_{\dot\alpha}^+$ obey the anticommutation relation (3.3).

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